A Level Set-Based Topology Optimization Method
For Three-Dimensional Acoustic Problems
Using Fast Multipole Boundary Element Method

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1. Abstract
This paper presents a level set-based topology optimization method for three-dimensional acoustic problems using Boundary Element Method and Fast Multipole Method. First, Fast Multipole Boundary Element Method for acoustic problems is formulated. Second, the level set-based topology optimization method using Tikhonov regularization method is briefly discussed. Next, we construct an optimization algorithm which uses the Fast Multipole Boundary Element Method to solve equilibrium equation and Finite Element Method to update the level set function. Finally, to confirm the validity and usefulness of the proposed topology optimization method, an optimum design example is provided.

2. Keywords: Topology Optimization, Level Set Method, Boundary Element Method, Fast Multipole Method, Acoustic Problems.

3. Introduction
This paper presents a topology optimization method for three-dimensional acoustic problems using Fast Multipole Boundary Element Method (FM-BEM) and level set-based structural boundary expressions incorporating a fictitious interface energy model. Although design of acoustic structure is one of important factor in mechanical engineering, the design based on the trial and error approach has some difficulty to obtain appropriate acoustic performance. To drastically overcome the issue, topology optimization methods for acoustic problem based on the Finite Element Method (FEM) are proposed [1]. However, in exterior acoustic problems, a lot of finite elements are required, since the approximated infinite space has to be solved. In contrast the FEM, the Boundary Element Method (BEM) is only required boundary elements to solve exterior acoustic problems. Moreover, to solve fast the boundary value problem, FM-BEM are proposed based on the Fast Multipole Method (FMM) and the BEM. On the other hand, level set-based shape optimization methods [2, 3] are proposed as a new type of structural optimization method and applied to several optimization problem, such as vibration problems, optimum design of compliant mechanisms and multiphysics actuators. The level set-based shape optimization methods offers a number of advantages when compared with conventional topology optimization methods, such as the Solid Isotropic Material with Penalization (SIMP) method, density approaches and Homogenization Design Method (HDM), namely, clear structural boundary expressions and free from grayscales in the obtained optimal configurations. These methods are classified to shape optimization method, since the level set function is updated using Hamilton-Jacobi equation which derived form boundary advection concept. Recently, Yamada et al. proposed level set-based topology optimization method is proposed [4]. In this method, a fictitious interface energy model is introduced and the level set function is updated using a reaction-diffusion equation. The geometrical complexity of the obtained optimal configurations can be controlled by adjusting a regularization factor. In this paper, first, the level set-based topology optimization method and the FM-BEM for acoustic problem is briefly discussed. Next, an acoustic topology optimization problem considering infinite space is formulated. Third, based on the level set boundary expressions, an numerical algorithm for generating boundary elements is constructed. Based on the mesh generating algorithm, the formulation and FM-BEM, an optimization algorithm is constructed. Finally, several three-dimensional numerical example problem is shown to confirm validity and utility of proposed

1
where \( p(x) \) is sound pressure, \( k = \omega/c \) is wave number, \( \omega \) is angular frequency, \( f(x) \) is sound source term, \( c \) is sound velocity, \( n \) is unit normal vector which direction is outside, \( \rho \) is medium density and \( v(x) \) is particle velocity. Then, the boundary integral equation is derived as following,

\[
c(x)p(x) = - \int_{\Gamma} q^*(x, y) p(y) d\Gamma_y + \int_{\Gamma} p^*(x, y) q(y) d\Gamma_y + \int_{\Omega} p^*(x, y) f(y) d\Omega_y,
\]

where \( p^*(x, y) \) is a fundamental solution of the Helmholtz equation, \( q^*(x, y) \) is a normal component of the gradient of the fundamental solution at point \( y \) located on the boundary and \( c(x) \) is defined as follows:

\[
\begin{align*}
  c(x) &= 1 \quad \text{in } \Omega \\
  c(x) &= \frac{1}{2} \quad \text{on } \Gamma \\
  c(x) &= 0 \quad \text{in } \bar{\Omega}
\end{align*}
\]

In three dimensional problems, the fundamental solution \( p^*(x, y) \) and the normal component of the gradient \( q^*(x, y) \) is derived as following:

\[
\begin{align*}
  p^*(x, y) &= \frac{e^{ikr}}{4\pi r} \\
  q^*(x, y) &= \frac{-1 + ikr}{4\pi r^2} \frac{\partial r}{\partial n(y)} e^{ikr},
\end{align*}
\]

where \( r = |x - y| \). The sound pressure gradient is calculated using following equation.

\[
c(x) \frac{\partial p(x)}{\partial x_i} = - \int_{\Gamma} \frac{\partial q^*(x, y)}{\partial x_i} p(y) d\Gamma_y + \int_{\Gamma} \frac{\partial p^*(x, y)}{\partial x_i} q(y) d\Gamma_y + \int_{\Omega} \frac{\partial p^*(x, y)}{\partial x_i} f(y) d\Omega_y
\]

4.2. Fast Multipole Method

Using the fast multipole expansion, the fundamental solution (6) is replaced with following equation.

\[
\frac{e^{ikr}}{4\pi r} = \frac{i k}{4\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1)O_n^m(k, \vec{Ox}) I_n^m(k, \vec{Oy}), \quad |\vec{Ox}| > |\vec{Oy}|,
\]

The notations are defined as,

\[
\begin{align*}
  O_n^m(k, \vec{Ox}) &= h_n^{(1)}(k|\vec{Ox}|) Y_n^m(\vec{Ox}) \\
  I_n^m(k, \vec{Oy}) &= j_n(k|\vec{Oy}|) Y_n^m(\vec{Oy})
\end{align*}
\]

where \( j_n \) is a spherical Bessel function and \( h_n^{(1)} \) is a spherical Hankel function. In addition, \( Y_n^m \) is a spherical surface harmonic function, which is defined as,

\[
Y_n^m(\theta, \phi) = \sqrt{\frac{(n-m)!}{(n+m)!}} P_n^m(\cos \theta) e^{im\phi},
\]

where \( P_n^m \) is an associated Legendre function, \((\phi, \theta)\) is polar coordinate system of \( \vec{Ox} \).
Next, we apply the plane-wave-expansion method to the fundamental solution (9). As a result, the fundamental solution is represented as following integral on the unit spherical surface.

\[
\frac{e^{ikr}}{4\pi r} = \frac{ik}{16\pi^2} \int_{S_0} e^{-ik\hat{k} \cdot \vec{O}y} D(\hat{k}, \vec{O}x) dS_k
\]

(13)

\[\hat{k} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta),\]

(0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi),

(14)

where \(S_0\) is the unit spherical surface, \(\hat{k}\) is the unit directional vector and \(D(\hat{k}, \vec{O}x)\) is following diagonalized transformation matrix.

\[
D(\hat{k}, \vec{O}x) = \sum_{n=0}^{\infty} \frac{i^n (2n + 1) h_n^{(1)}(k|\vec{O}x|) P_n(\hat{k} \cdot \hat{O}x)}{2n + 1},
\]

(15)

where \(P_n\) is the Legendre polynomial.

In the wide-band fast multipole boundary element method, each fundamental solutions, that is equation (9) and equation (13), are used properly. The detail of the formulations and implementations are discussed in References [5, 6]

5. Optimization

5.1. Level Set-Based Topology Optimization Method

We briefly discuss the level set-based topology optimization method. First, following topology optimization problem is defined using the objective functional \(F\) and the constraint functional \(G\).

\[
\inf_{\chi(x)} F = \int_{\Omega} f(x)\chi(x) d\Omega
\]

subject to \(G = \int_{\Omega} \chi(x) d\Omega - V_{\text{max}} \leq 0,\)

(16)

(17)

where \(V_{\text{max}}\) is upper limit of the volume constraint and \(\chi\) is the characteristic function defined as follows:

\[
\chi(x) = \begin{cases} 
1 & \text{if } x \in \Omega \\
0 & \text{if } x \notin \Omega
\end{cases}
\]

(18)

Next, we introduce the level set function which represents structural boundaries using the iso-surface as shown in Figure 1. That is, the level set function \(\phi(x, y)\) is defined as follows:

Figure 1: Level set function \(\phi\)
\[
\begin{cases}
0 < \phi(x) \leq 1 & \text{for } \forall x \in D \setminus \Omega \\
\phi(x) = 0 & \text{for } \forall x \in \partial \Omega \\
-1 \leq \phi(x) < 0 & \text{for } \forall x \in \Omega \setminus D,
\end{cases}
\] (19)

where \( D \) is the fixed design domain. Using the level set-based structural boundary expressions, the topology optimization problem can be replaced with following optimization problem,

\[
\inf_{\phi(x)} F = \int_D f(x) \chi(\phi) d\Omega
\] (20)

subject to \( G = \int_D \chi(\phi) d\Omega - V_{\max} \leq 0 \). (21)

Topology optimization method must be regularized, since it is an ill-posed problem. In this paper, the topology optimization problem is applied the Tikhonov regularization method. That is, a regularization term is introduced to the objective functional.

\[
\inf_{\phi(x)} F_R = \int_D f(x) \chi(\phi) d\Omega + \int_D \frac{1}{2} |\nabla \phi|^2 d\Omega
\] (22)

subject to \( G = \int_D \chi(\phi) d\Omega - V_{\max} \leq 0 \), (23)

where \( F_R \) is an regularized objective functional. The topology optimization problem is replaced with solving a reaction-diffusion equation. The detail is discussed in the reference [4].

5.2. Formulation of Acoustic Optimization Problem
We consider following maximum sound pressure problem with volume constraint.

\[
\inf_{\phi} F = \mathcal{J}(p_{Re}, p_{Im})
\] (24)

subject to \( a(\phi, p_{Re}, p_{Im}, \tilde{p}_{Re}, \tilde{p}_{Im}) - b(\tilde{p}_{Re}, \tilde{p}_{Im}) = 0 \) for \( p \in U_p, \forall \tilde{p} \in U_p, G \leq 0 \) (25) (26)

where the notations defined as follows:

\[
p = p_{Re} + p_{Im}i, \quad \tilde{p} = \tilde{p}_{Re} + \tilde{p}_{Im}i
\] (27)

\[
\mathcal{J}(p_{Re}, p_{Im}) = \frac{1}{2} \int_{\Omega_o} (p_{Re} + p_{Im}i)(p_{Re} - p_{Im}i) d\Omega
\] (28)

\[
a(\phi, p_{Re}, p_{Im}, \tilde{p}_{Re}, \tilde{p}_{Im}) = \int_D \nabla p \cdot \nabla \tilde{p} \chi d\Omega + \int_D k^2 p \tilde{p} \chi d\Omega
\] (29)

\[
b(p_{Re}, p_{Im}) = \int_{\Omega_s} f \tilde{p} d\Omega
\] (30)

\[
U_p = \left\{ p \in H^1(D) \text{ with } \frac{\partial p}{\partial n} = 0 \text{ on } \Gamma \right\}
\] (31)

where \( \Omega_o \) is a fixed observation domain and \( \Omega_s \) is fixed sound source domain.

Next, the acoustic optimization problem is replaced with an optimization problem without constraints using Lagrange’s method of undetermined multipliers. That is, when the Lagrange is defined as \( \bar{F} \) and the Lagrange multiplier of the volume constraint is \( \lambda \), the acoustic optimization problem is replaced with the following,

\[
\inf_{\phi} \bar{F} = \mathcal{J}(p_{Re}, p_{Im}) + a(\phi, p_{Re}, p_{Im}, \tilde{p}_{Re}, \tilde{p}_{Im}) - b(\tilde{p}_{Re}, \tilde{p}_{Im}) + \lambda G(\phi).
\] (32)

In this paper, optimal structures will be obtained by solving the above acoustic optimization problem.

Next, the necessary optimality conditions for the above acoustic optimization problem are obtained as follows:

\[
\bar{F}' = 0, \quad a(\phi, p_{Re}, p_{Im}, \tilde{p}_{Re}, \tilde{p}_{Im}) - b(\tilde{p}_{Re}, \tilde{p}_{Im}) = 0, \quad \lambda G(\phi) = 0, \quad \lambda \geq 0, \quad G(\phi) \leq 0
\] (33)
The level set function representing the optimal structures satisfies the above KKT conditions. That is, the optimal solution must satisfy the above Equations (33). Conversely, solutions obtained by Equations (33) are optimal solution candidates, however obtaining this level set function directly is problematic. Therefore, the optimization problem becomes a problem of solving time evolutionary equation, which will provide an optimal solution candidate.

5.3. Time Evolutionary Problem
In this section, a time evolutionary equation for updating the level set function is derived. First, a fictitious time $t$ is introduced, and it is assumed that the level set function $\phi$ is also a function of $t$, to represent changing the configurations over time. We also assume that the degree to which the level set function changes is proportional to the gradient of the Lagrangian with regularization term $\bar{F}_R$, as shown in the following:

$$\frac{\partial \phi}{\partial t} = -K^2 \bar{F}_R = -K^2 (F^T - \tau \nabla^2 \phi),$$

(34)

where $K$ is a constant of proportion. The level set function is updated using above reaction-diffusion equation.

5.4. Optimization Algorithm
Figure 2 shows the flowchart of the optimization procedure. As shown, the initial configuration is first initialized the level set function $\phi(x)$.

Figure 2: Flowchart of optimization procedure

set. In the second step, boundary mesh is generated based on the level set function using marching cube, which is defined finite element for updating the level set function. Based on the generated boundary elements, the governing equations are solved using the FM-BEM. Next, the objective functional computed. Here, the topology optimization process is finished of the objective functional converged, otherwise the sensitivities are computed using adjoint variable method. Next, the level set function $\phi(x)$ is updated based on the reaction-diffusion equation (34) using the finite element method.

6. Numerical Example
A numerical example is now presented to confirm the utility and validity of proposed topology optimization method. As shown Figure 3, the fixed design domain is a rectangular solid which size is 4.0m×4.0m×0.4m and is discretized using a structural finite mesh for updating level set function. The
sound source which wavelength is \( \lambda = 1.9\text{[m]} \) and observation domain are located at \((2.0\text{[m]}, 2.0\text{[m]}, 2.0\text{[m]})\) and \((2.0\text{[m]}, 1.0\text{[m]}, 2.0\text{[m]})\), respectively. The regularization parameter \( \tau \) is \(1.0 \times 10^{-4}\) and upper limit of the volume constraint is 50\% of the fixed design domain. Figure 4 shows obtained optimal configuration.

![Figure 3: Design domain](image)

As shown, the obtained optimal configuration is clear configuration, since the boundary mesh is generated based on the level set function.

### 7. Conclusions

This paper proposed a level set-based topology optimization method for three-dimensional acoustic problems using the BEM and the FMM. We achieved the following:

1. A topology optimization method for three-dimensional acoustic problem was constructed, based on the level set method and the fast multipole boundary element method. In this method, boundary mesh is generated based on the level set function at each optimization step.
2. The presented numerical example confirmed that the proposed method can be obtained optimal configuration.

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### References


Figure 4: Optimal configuration
