Study on transition from photonic-crystal laser to random laser

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Abstract: The dependence of the lasing threshold on the amount of positional disorder in photonic crystal structures is newly studied by means of the finite element method, not of the finite difference time domain method usually used. A two-dimensional model of a photonic crystal consisting of dielectric cylinders arranged on a triangular lattice within a circular region is considered. The cylinders are assumed to be homogeneous and infinitely long. Positional disorder of the cylinders is introduced to the photonic crystals. Optically active medium is introduced to the interspace among the cylinders. The population inversion density of the optically active medium is modeled by the negative imaginary part of dielectric constant. The ratio between radiative power of electromagnetic field without amplification and that with amplification is computed as a function of the frequency and the imaginary part of the dielectric constant, and the threshold of the imaginary part, namely population inversion density for laser action is obtained. These analyses are carried out for various amounts of disorder. The variation of the lasing threshold from photonic-crystal laser to random laser is revealed by systematic computations with numerical method of reliable accuracy for the first time. Moreover, a novel phenomenon, that the lasing threshold have a minimum against the amount of disorder, is found. In order to investigate the properties of the lasing states within the circular system, the distributions of the electric field amplitudes of the states are also calculated.

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References and links

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frequencies. Random lasers occur from multiple scatterings and interference effects, causing
realization in photonic crystals [20–22] due to the extremely low group velocities at band edge
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1. Introduction

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Anderson localizations [23], in disordered structures, thus their laser modes take various and
complex forms. Lawandy et al. mentioned that the lasing threshold of random lasers was ex-
tremely low [1].

In our previous study [24, 25], we discussed lasing threshold of random lasers. The effects of positions of active medium [24] and filling factor of dielectric cylinders [25] on the lasing threshold were investigated and appropriate dielectric structures for lower threshold laser action were proposed.

Photonic crystals including small amounts of disorders have been investigated actively. There are many numerical and experimental studies on the effects of disorders on the optical properties of photonic crystals, such as band structures [26–28], light localizations [26, 29, 30], transmissions [28, 31–40], and reflectance [41]. These previous studies treated various types of disorders, for example, in size [32, 38, 40, 42–44], positions [5, 7–9, 33, 36, 39, 42, 45–47], shapes of dielectric material [26, 48, 49], refractive indices [34, 38, 43], surface roughness [33, 36, 46, 47, 50–52], and fabrication errors [53, 54]. Additionally, we can also find a paper showing the effects of disorder on lasing phenomena in dielectric structures. Kwan, et al. [55] studied the effect of position and size-disorder on the emission spectra, and the lasing frequency at the highest intensity emission with a fixed pumping rate. However, no studies are found on the influence of disorder on lasing threshold.

In the present study, we study the transition from photonic-crystal laser to random laser as a new research topic to realize lower threshold laser action. We simulate laser actions numerically to investigate the changes of lasing thresholds under the influence of disorders in two-dimensional dielectric structures. Disorder of dielectric structures is treated by giving positional disorders of dielectric cylinders. The amount of disorder is parameterized with the length between the grid points corresponding to the centers of fictitious cylinders distributed periodically and the centers of randomly distributed cylinders. We reveal the change of lasing threshold against the amount of disorder and find an appropriate one for lower-threshold laser action.

2. Analysis model

We show a model of a dielectric system in Fig. 1(a). Dielectric cylinders are assumed to be infinitely long in vertical direction (z-direction) and light waves propagate within xy-plane. Figure 1(b) shows the concept of a random system from the top view. Dielectric cylinders

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Fig. 1. Concept of random media. (a) A random structure model. (b) An illustration of whole random system.
are arranged randomly in the region between circles $C_{\text{in}}$ and $C_{\text{g}}$, as shown in Fig. 1(b). An oscillating polarization is assumed to exist at the center of the circle $C_{\text{in}}$ as a light source. Radii of $C_{\text{in}}$ and $C_{\text{g}}$ are denoted by $R_{\text{in}}$ and $R_{\text{g}}$, respectively. We compute the fluxes of Poynting vectors of out-flowing light waves on the circle $C_{\text{out}}$ whose radius is $R_{\text{out}}$. The unit outward normal vectors to $C_{\text{in}}$ and $C_{\text{out}}$ are denoted by $n_{\text{in}}$ and $n_{\text{out}}$, respectively. We define three regions: $\Omega_{\text{act}}$, $\Omega_{\text{cylinder}}$ and $\Omega_{\text{out}}$, where $\Omega_{\text{act}}$ is in the interspace among the cylinders inside the circle $C_{\text{out}}$, $\Omega_{\text{cylinder}}$ is the union of the regions inside the cylinders, and $\Omega_{\text{out}}$ is the region outside the circle $C_{\text{out}}$. The optically active materials are assumed to be filled in the region $\Omega_{\text{act}}$.

2.1. Model parameters

In Table 1 is shown the parameters used to create the analysis models. The radii of the cross sections of the cylinders are assumed to be the same, and the radius, $a$, is treated as the characteristic length. The size of the analysis models is normalized by $a$. We create the analysis model of the periodic structure of triangular lattice, by giving the transfer mean free path (TMFP), denoted by $l$, and the periodic length of the periodic structure. The values of the TMFP and the periodic length are given as $l = 1.47735a$ and $3.47735a$, respectively. The coordinate values of the cylinder’s center are specified in single precision numbers.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of cylinders</td>
<td>$a$</td>
<td>$1$ : characteristic length</td>
</tr>
<tr>
<td>Filling factor</td>
<td>$f$</td>
<td>$0.3$ (30%)</td>
</tr>
<tr>
<td>Radius of $C_{\text{in}}$</td>
<td>$R_{\text{in}}$</td>
<td>$1.5a$</td>
</tr>
<tr>
<td>Radius of $C_{\text{g}}$</td>
<td>$R_{\text{g}}$</td>
<td>$39a$</td>
</tr>
<tr>
<td>Radius of $C_{\text{out}}$</td>
<td>$R_{\text{out}}$</td>
<td>$40a$</td>
</tr>
<tr>
<td>TMFP</td>
<td>$l = a \left( \sqrt{2\pi/\sqrt{3}f} - 2 \right)$</td>
<td>$1.47735a$</td>
</tr>
<tr>
<td>Periodic length</td>
<td></td>
<td>$3.47735a$ $(= l + 2a)$</td>
</tr>
<tr>
<td>Position of center</td>
<td>$(x_0, y_0)$</td>
<td>$(0, 0)$</td>
</tr>
<tr>
<td>Width of PML</td>
<td></td>
<td>$3a$</td>
</tr>
<tr>
<td>Width of whole model</td>
<td></td>
<td>$93a$</td>
</tr>
<tr>
<td>The number of cylinders</td>
<td></td>
<td>$432$</td>
</tr>
</tbody>
</table>

2.2. Disordered systems

A parameter used to control disorder of the cylinder arrangement is defined in Fig. 2. The circles drawn by broken lines illustrate fictitious cylinders of a periodic structure. The centers of these circles are used as the datum points to control the cylinders arranged in disorder. Centers of the circles are denoted by $x_p$. A circle drawn by a solid line illustrates a cylinder arranged in disorder, whose center is denoted by $x_r$. The disordered position $x_r$ is determined by a sum of $x_p$ and a random vector $\Delta x_r$, as follows:

$$x_r(n, m) = x_p(n, m) + \Delta x_r,$$  \hspace{1cm} (1)

where $n$ and $m$ are lattice-point numbers and $(n, m) \neq (0, 0)$. $x_p$ are defined as

$$x_p(n, m) = nr_1 + mr_2,$$  \hspace{1cm} (2)

where $r_1$ and $r_2$ are lattice vectors defined as

$$r_1 = \left[ 3.47735a \times \sqrt{3}/2, 3.47735a \times 1/2 \right]^T,$$  \hspace{1cm} (3)
where \( \Delta x_r \) is the maximum length of the random vector. The amounts of disorders in dielectric structures are evaluated with \( |\Delta x_r|_{\text{max}} / a \), that is, the maximum length normalized by the radii of cylinders. In the case \( |\Delta x_r|_{\text{max}} / a = 0.00 \), a dielectric structure becomes periodic. In Fig. 2, \( L_H \), the distance between the edge of the hexagonal lattice and the center of the cylinder periodically distributed and included in the lattice, is equal to 1.73867\( a \), a half of the periodic length. Therefore, when \( L_H < |\Delta x_r|_{\text{max}} \), the center of the cylinder distributed randomly is located within the adjacent hexagonal lattice. When \( |\Delta x_r|_{\text{max}} \) is smaller than \( L_H - a = 0.73867a \), the entire region of the cylinder is included in each lattices.

\[
\mathbf{r}_2 = [0.0, 3.47735a]^T,
\]

where 3.47735\( a \) is the periodic length.

We restrict the length of the random vector \( \Delta x_r \) as

\[
0 \leq |\Delta x_r| \leq |\Delta x_r|_{\text{max}}
\]

where \( |\Delta x_r|_{\text{max}} \) is the maximum length of the random vector. The amounts of disorders in dielectric structures are evaluated with \( |\Delta x_r|_{\text{max}} / a \), that is, the maximum length normalized by the radii of cylinders. In the case \( |\Delta x_r|_{\text{max}} / a = 0.00 \), a dielectric structure becomes periodic. In Fig. 2, \( L_H \), the distance between the edge of the hexagonal lattice and the center of the cylinder periodically distributed and included in the lattice, is equal to 1.73867\( a \), a half of the periodic length. Therefore, when \( L_H < |\Delta x_r|_{\text{max}} \), the center of the cylinder distributed randomly is located within the adjacent hexagonal lattice. When \( |\Delta x_r|_{\text{max}} \) is smaller than \( L_H - a = 0.73867a \), the entire region of the cylinder is included in each lattices.
We analyze lasing phenomena in dielectric structures for various $|\Delta x_r|_{\text{max}}$. The analysis models and radial distribution functions for these values are shown in Figs. 3 and 4, respectively. Radial distribution functions describe the change of the density of dielectric cylinders as the function of the distance from the center of dielectric structures. The functions express how dielectric structures are disordered.

3. Formulation

In this study, we simulate lasing phenomena in dielectric structures consisting of homogeneous cylinders in the case of TM mode by using the node-base FEM. We use perfectly matched layers (PMLs) [56] to simulate scatterings in the open region, and employ an optimized absorbing function [57–59] that minimizes numerical reflections.

3.1. Basic equations of electromagnetic scattering problem

We assume an electric polarization oscillating with angular frequency $\omega$ at the center of the entire region of the random system, $x_0$, as a light source (Fig. 1). The electric and magnetic fields, $E$ and $H$, are assumed to be time-harmonic waves with the same angular frequency as that of the light source. The following non-homogeneous equation is derived from Maxwell’s equations as

$$\nabla \times [\nabla \times E(x)] - \frac{\omega^2}{c^2} \varepsilon(x) E(x) = \frac{\omega^2}{\varepsilon_0 c^2} D_d \delta(x - x_0), \tag{6}$$

where $c$ is the speed of light in vacuum, $\varepsilon_0$ and $\varepsilon(x)$ are the permittivity in vacuum and the position-dependent relative permittivity, $D_d$ is the polarization vector, $\delta(x)$ is Dirac’s delta function.

We define total electric field $E(x)$ as the sum of the scattering and incident fields as follows:

$$E(x) = E_s(x) + E_i(x), \tag{7}$$

where, $E_i(x)$ is the electric field in the region without scatterers, satisfying the following equation:

$$\nabla \times [\nabla \times E_i(x)] - \frac{\omega^2}{c^2} \varepsilon_i E_i(x) = \frac{\omega^2}{\varepsilon_0 c^2} D_d \delta(x - x_0), \tag{8}$$

where $\varepsilon_i$ is the constant relative permittivity in $\Omega_{\text{out}}$ (Fig. 1). We substitute Eqs. (7) and (8) to Eq. (6), to have

$$\nabla \times [\nabla \times E_s(x)] - \frac{\omega^2}{c^2} \varepsilon(x) E_s(x) = \frac{\omega^2}{c^2} \left[ \varepsilon(x) - \varepsilon_i \right] E_i(x). \tag{9}$$

When we assume TM mode, the incident field $E_i(x)$ satisfying Eq. (8) can be expressed by 0th-order Hankel function of the first kind, $H_0^{(1)}$, as follows:

$$E_i(x) = \frac{\omega^2}{\varepsilon_0 c^2} D_d \frac{i}{4} H_0^{(1)} \left( \frac{\omega}{c} \sqrt{\varepsilon_i} |x - x_0| \right), \tag{10}$$

where $i$ is the imaginary unit. We solve Eq. (9) by using FEM formulated based on Galerkin’s method and obtain the total electric field $E(x)$ defined by Eq. (7).

3.2. Population inversion density of optically active materials

Population inversion density of optically active material can be modeled by a negative imaginary part of relative permittivity, $-\gamma$ ($\gamma > 0$) [20–22]. $\gamma$ is a parameter proportional to the...
population inversion density of an optically active material. Hence, \( \gamma \) at which a laser action occurs is interpreted as the threshold for the laser action [20–22]. We assume a system whose interspace among dielectric cylinders are filled with an optically active material, and set the imaginary part of relative permittivity in interspace among cylinders to \(-\gamma\). The relative permittivities in individual regions (Fig. 1) are given as follows:

\[
\varepsilon(x) = \begin{cases} 
1.0 + i(-\gamma) & \text{if } x \in \Omega_{\text{act}} \\
4.0 & \text{if } x \in \Omega_{\text{cylinder}} \\
1.0 & \text{if } x \in \Omega_{\text{out}}
\end{cases}
\]  

(11)

In the above modeling of active material, uniform gain media are assumed because the excitation by an external source is considered. We consider the population inversion as an independent parameter of electric field. The above parameterization of the population inversion may become well-approximated the model when the parameter corresponds to the population inversion in localization space in the case of spatially localized modes. In the case of spatially extended modes, the approximation becomes more appropriate because electric field is homogeneously extended. We can consider such a simple modeling of population inversion as the zeroth-order approximation.

### 3.3. Definition of amplification factor

Because of the assumption that electric and magnetic waves are time-harmonic, we need to compute Poynting vectors in the following time-averaged form:

\[
\langle S \rangle = \text{Re} \left( \frac{E \times H^*}{2} \right)
\]  

(12)

where \( \langle S \rangle \) is the time-average of Poynting vector \( S \), \( \text{Re}(Z) \) means the real part of complex vector \( Z \), and \( H^* \) denotes the complex conjugate of the magnetic field.

We define the amplification factor \( A \) by the ratio of the fluxes of the Poynting vectors of light, flowing out from the dielectric system, between the excited state \((\gamma > 0)\) and non-excited state \((\gamma = 0)\), as follows:

\[
A = \frac{\int_{C_{\text{out}}} \langle S \rangle \cdot \mathbf{n}_{\text{out}} \, dl \big|_{\gamma > 0}}{\int_{C_{\text{out}}} \langle S \rangle \cdot \mathbf{n}_{\text{out}} \, dl \big|_{\gamma = 0}}.
\]  

(13)

The light flux is calculated by a line integral of the Poynting vector along the circle \( C_{\text{out}} \).

### 3.4. Finite element analysis

In most numerical analyses of random lasers, lasing phenomena are simulated using finite-difference time-domain (FDTD) method. FDTD is effective for numerical analyses of electromagnetic (EM) waves in homogeneous optical media. In the analyses of EM waves in heterogeneous ones, FDTD simulations suffer from staircasing errors [60–62] due to spatial discretizations with square grids which are unfitted to curved interfaces between different optical materials. On the other hand, FEM, one of the most effective numerical methods for the analyses of EM waves, can model arbitrary shaped fields and create their mesh very easily. Therefore, staircasing errors may not arise if FEM is used in the analyses of random lasers because the FEM meshes are well-fitted to the interfaces.

Figure 5 illustrates the finite elements of an analysis model. A blue circular area in Fig. 5(a) shows a cross section of a dielectric cylinder. The edge size of the elements is approximately given as 0.08a. The computational accuracy is reduced due to the rotation of the electric field, \( \nabla \times \mathbf{E} \), which is needed to compute the complex conjugate of magnetic field, \( \mathbf{H}^* \) in Eq. (12).
Hence, we discretize the regions in the neighborhood of the circle $C_{\text{out}}$ into more smaller elements $0.01a$. Since the absorbing function of PML is nonlinear, it is difficult to integrate the components of the stiffness matrix for this region analytically. Therefore, numerical integration scheme, Gauss-Legendre quadrature formula, is employed to evaluate numerically the integrals including the absorbing function for the area discretized with the quadrilateral elements shown in Fig. 5(b). The numbers of nodes and elements are approximately one million and two million, respectively.

4. Results

4.1. Lasing frequency

Figure 6 shows the result of laser action in the periodic structure corresponding to $|\Delta x_r|_{\text{max}}/a = 0.00$. We compute the amplification factor for the frequency range $0.1 \leq \omega a/2\pi c \leq 0.3$ with fixed population inversion density $\gamma = 0.002$ to investigate lasing frequencies of the periodic structure corresponding to $|\Delta x_r|_{\text{max}}/a = 0.00$. We also show the band structure of the periodic structure, computed by the plane wave expansion method, in Fig. 7. In the result shown in Fig. 6, we cannot achieve sufficient computational accuracy in red-colored frequency ranges corresponding to band gaps in Fig. 7.

We observe laser action in the frequency range colored blue in Fig. 6, $0.222 \leq \omega a/2\pi c \leq 0.253$. 

Fig. 5. Finite elements. (a) Mesh in cylinder. (b) Mesh in PML.

Fig. 6. Amplification factor versus normalized frequency $\omega a/2\pi c$ for $\gamma = 0.002$. 

Fig. 7. Band structure of the periodic structure.
Fig. 7. Dispersion relation of light waves in the periodic structure corresponding to $|\Delta x_r|_{\text{max}}/a = 0.00$.

![Graph showing dispersion relation](image)

Fig. 8. Electric intensity distributions corresponding to the wave vector, $k = (3\pi/4a, \sqrt{3}\pi/4a)$ in the unit cell of the periodic structure. (a) The unit cell of the periodic structure. (b) Lowest frequency band. (c) 2nd lowest frequency band. (d) 3rd lowest frequency band. (e) 4th lowest frequency band. (f) 5th lowest frequency band.

![Image of electric intensity distributions](image)

0.240. We observe this frequency range, in which laser action occurs, corresponds to that of fourth-lowest band in Fig. 7. We show in Fig. 8 the electric field intensity distributions corresponding to the wave vector, $k = (3\pi/4a, \sqrt{3}\pi/4a)$, of each band in Fig. 7. Figure 8(a) shows the unit cell of the periodic structure. We find in the distribution of the fourth-lowest band shown in Fig. 8(e), the electric field intensity in interspace among dielectric cylinders becomes high, i.e., the electric field becomes intensive within an optically active material. The laser action in Fig. 6 is thought to be caused by such light localization in an optically active material.

### 4.2. Laser action

We compute the amplification factors $A$ defined by Eq. (13) for the ranges of $0.222 \leq \omega a/2\pi c \leq 0.240$ and $0.000 \leq \gamma \leq 0.021$, corresponding to the fourth lowest band. Figure 9 shows the laser action in the dielectric structures for each value of the disorder index $|\Delta x_r|_{\text{max}}/a$ normalized by $a$. We calculate the amplification factors for 127,041 grid points for $\omega a/2\pi c$ and $\gamma$, by dividing
Fig. 9. Amplification factor $A$ versus normalized frequency $\omega a/2\pi c$ and $\gamma$ for each $|\Delta x_r|_{\text{max}}/a$. (a) $|\Delta x_r|_{\text{max}}/a = 0.00$ (periodic structure). (b) $|\Delta x_r|_{\text{max}}/a = 0.0625$. (c) $|\Delta x_r|_{\text{max}}/a = 0.125$. (d) $|\Delta x_r|_{\text{max}}/a = 0.250$. (e) $|\Delta x_r|_{\text{max}}/a = 0.500$. (f) $|\Delta x_r|_{\text{max}}/a = 1.00$. (g) $|\Delta x_r|_{\text{max}}/a = 2.00$. (h) $|\Delta x_r|_{\text{max}}/a = 4.00$. (i) A peak at the bottom edge frequency $\omega a/2\pi c = 0.22196531$ of the 4th band for $|\Delta x_r|_{\text{max}}/a = 0.00$ (periodic structure).

$\omega a/2\pi c$ and $\gamma$ directions uniformly into 900 an 140 intervals, respectively, and seek the steep peaks of the surface of amplification factor $A$.

Figure 9(a) shows the distribution of the amplification factor corresponding to the combinations of $\omega a/2\pi c$ and $\gamma$. From this result, we find specific combinations of the values of $\omega a/2\pi c$ and $\gamma$ at which lasing phenomena occur in the periodic structure. Excitation within the fourth lowest band in the dispersion relation occurs as the surface of the amplification factor $A$ and the lasing phenomena occur at their peaks. Figure 9(i) shows a laser action occurring at band-edge frequency $\omega a/2\pi c = 0.22196531$ of the 4th band for $|\Delta x_r|_{\text{max}}/a = 0.00$ (periodic structure).

In Figs. 9(b) to 9(h), we observe that surface of amplification factor reflects the effect of disorder on laser action. The lasing phenomena caused by random light scatterings noticeably emerge as the disorder of the structure increases. Such lasing phenomena are found more in higher frequency range $0.23 \leq \omega a/2\pi c \leq 0.24$, in the results for small amounts of disorder, $|\Delta x_r|_{\text{max}}/a = 0.0625, 0.125$, and 0.250, shown in Figs. 9(b), 9(c), and 9(d), respectively. Laser actions occurring in the periodic structure are also found even in the results of small amounts of disorder. However, in the case of higher disorder, i.e. $0.500 \leq |\Delta x_r|_{\text{max}}/a$, lasing phenomena caused by random light scatterings become noticeable also in the lower frequency range, and lasing phenomena found in the periodic structure disappear.

4.3. Lasing threshold

Parameter $\gamma$, interpreted as the imaginary part of relative permittivity, is proportional to population inversion density of optically active material. Hence, steep peaks of the amplification factors in the region with small $\gamma$ are interpreted as occurrences of low-threshold laser genera-
Fig. 10. Relation between normalized disorder index |Δx_r|_{max}/a and the average value of lasing thresholds, namely, lowest γ with error bar.

Fig. 11. Electric amplitude distributions of lasing states with tight confinement (sample 1) 
(a) |Δx_r|_{max}/a = 0.00, ωa/2πc = 0.22196531, γ = 0.0004965. (b) |Δx_r|_{max}/a = 0.0625, ωa/2πc = 0.2221812, γ = 0.0007350. (c) |Δx_r|_{max}/a = 0.125, ωa/2πc = 0.2223220, γ = 0.001140. (d) |Δx_r|_{max}/a = 0.250, ωa/2πc = 0.222716, γ = 0.002490.

tion. To find how lasing threshold changes in accordance with the increase of the disorder index |Δx_r|_{max}, we investigate the smallest value of γ at which laser action occurs. To seek the more precise values of ωa/2πc and γ at which the lowest-threshold laser action occurs, we scan the neighborhood of the amplification peak using finer numerical steps.

Average values of the lowest thresholds for different disorder index values are plotted in Fig. 10 with error bars showing the ranges of γ. We analyze 10 different types of cylinder arrangement for each disorder index other than |Δx_r|_{max}/a = 0.00 to check the lasing threshold tendency. We find two types of lowest-threshold laser modes: those with tight confinement and spacial extension of light wave.

The threshold of the tightly confined mode, plotted by red squares and line in Fig. 10, rises as the disorder index increases. Such laser modes finally disappear when 0.500 ≤ |Δx_r|_{max}/a.

We show next the electric amplitude distributions (EADs) of tightly confined laser mode in Fig. 11. The angular distributions of group velocities of light waves in a periodic array are computed with tight-binding approximation and are shown in Fig. 12. From the comparison between the results shown in Figs. 11 and 12, we confirm that such tightly confined modes occur from the periodic array structure of dielectric cylinders. We observe that light wave confinement
becomes weak as the disorder index increases, particularly in Fig. 11(d). Such leakage of light is considered to lead to the rise in the lasing threshold of the laser action with the tight confinement of light wave.

We also show the lowest threshold lasing phenomena with spacial extension of light wave in Fig. 10. Blue triangles and line in the figures show the dependence of the threshold of the lasing phenomena with spatial extension on the disorder index. As this index increases, the threshold rises in $|\Delta x_r|_{\text{max}}/a < 0.250$, then decreases in $0.250 < |\Delta x_r|_{\text{max}}/a < 1.00$. In Fig. 10, the average threshold of 10 different cylinders arrangements rises again in $1.00 < |\Delta x_r|_{\text{max}}/a$. The average threshold becomes minimum at $|\Delta x_r|_{\text{max}}/a = 1.00$.

We show in Fig. 13 the EADs of the spatially extended modes. We observe in Figs. 13(a), 13(b), and 13(c) the collapse of the localization forms from that of the periodic structure as the disorder index increases. Lasing threshold of the laser action with spatial extension is found to decrease in $0.250 < |\Delta x_r|_{\text{max}}/a < 1.00$. In view of the EADs shown in Figs. 13(d) and 13(f), such decrease of lasing threshold is caused by increase in multiple scatterings.

We observe the average of lasing threshold becomes locally minimum at $|\Delta x_r|_{\text{max}}/a = 1.00$, and another increase of the lasing threshold for $1.00 < |\Delta x_r|_{\text{max}}/a$ in Fig. 10. We compute mode volumes of the lasing peaks oscillating at the smallest $\gamma$ in each sample to investigate how the optical properties of the laser mode affect on lasing threshold. The mode volume is defined as follows:

$$V = \frac{\int_{\Omega_{\text{act}}} + \Omega_{\text{cylinder}} \varepsilon(x) |E(x)|^2 d\Omega}{\max \left[ \varepsilon(x) |E(x)|^2 \right] \lambda^2},$$

where $\lambda = 2\pi c/\omega$ is the wavelength in vacuum. Figure 14 shows the relation between the mode volume and the amount of positional disorder. The mode volume decreases as the amount of disorder increases in the state of spatially extended modes. In the case of tightly confined modes, the mode volume increases because the periodic structure is disordered and the increase indicates leaks of the confined light caused by disorder. The average values of the mode volumes of 10 samples in $0.75 \leq |\Delta x_r|_{\text{max}}/a \leq 1.5$ become smaller than those in $2 \leq |\Delta x_r|_{\text{max}}/a \leq 4$.

Based on the comparison between Figs. 10 and 14, we find that lasing threshold tends to become lower as the mode volume becomes smaller. The above comparison indicates that strong light confinements are needed in order that low-threshold random lasings are realized.

When shifting the positions of the cylinders using the random numbers, there is a possibility that the shifted cylinder may overlaps for $0.73867 \leq |\Delta x_r|_{\text{max}}/a$. In such a case, the random number value is discarded and a regenerated random number is used to determine the shifted position of the cylinder. Therefore, the average of the random numbers employed to generate the random arrangement of the cylinders may differ from 0.5 of the admissible shift range.

![Fig. 12. Polar plots of the group velocity of tight-binding model. (a) $\omega a/2\pi c = 0.3650$. (b) $\omega a/2\pi c = 0.3675$.](image)
Fig. 13. Electric amplitude distributions of lasing states with spatial extension (sample 1). (a) $|\Delta x_r|_{\text{max}}/a = 0.00$, $\omega a/2\pi = 0.2310212$, $\gamma = 0.001380$. (b) $|\Delta x_r|_{\text{max}}/a = 0.125$, $\omega a/2\pi = 0.2308372$, $\gamma = 0.001470$. (c) $|\Delta x_r|_{\text{max}}/a = 0.125$, $\omega a/2\pi = 0.2308372$, $\gamma = 0.001470$. (d) $|\Delta x_r|_{\text{max}}/a = 0.500$, $\omega a/2\pi = 0.2297800$, $\gamma = 0.001785$. (e) $|\Delta x_r|_{\text{max}}/a = 0.750$, $\omega a/2\pi = 0.2280060$, $\gamma = 0.001575$. (f) $|\Delta x_r|_{\text{max}}/a = 1.00$, $\omega a/2\pi = 0.2238840$, $\gamma = 0.001515$. (g) $|\Delta x_r|_{\text{max}}/a = 1.25$, $\omega a/2\pi = 0.2276240$, $\gamma = 0.001995$. (h) $|\Delta x_r|_{\text{max}}/a = 4.00$, $\omega a/2\pi = 0.2321304$, $\gamma = 0.002430$.

Fig. 14. Mode volume versus normalized disorder index $|\Delta x_r|_{\text{max}}/a$.

Figure 15 shows the averages of $|\Delta x_r|_{\text{max}}/a$ of all samples analyzed in the present study. The average becomes 0.5 theoretically if all the generated random numbers are used to rearrange the cylinders because the distributions of $|\Delta x_r|$ are given by random numbers of a uniform distribution between 0 and 1. In the range $0.73867 \leq |\Delta x_r|_{\text{max}}/a \leq 2.00$, the effect of discard of some random numbers are observed clearly. The average values tend to decrease as $|\Delta x_r|_{\text{max}}$ increases.

The serious influence occurs when the average of $|\Delta x_r|$ corresponds to 0.73867$a$. Considering
the average of the random number \( m = 0.5 \), its dispersion \( \rho = 1/12 \), and the standard deviation \( \sigma = \sqrt{1/12} \), we find that the influence of discarding some of the random numbers occurs for

\[
(m - \sigma)|\Delta x_r|_{\text{max}} < 0.73867a < (m + \sigma)|\Delta x_r|_{\text{max}},
\]

resulting in

\[
0.93660 < |\Delta x_r|_{\text{max}}/a < 3.49544.
\]

This range agrees well with the results shown in Fig. 15. Hence, the local minimum of the lasing threshold observed in Fig. 10 may be the result of the employment of non-uniform random numbers to avoid the overlapping to the cylinders when shifting them from the hexagonal grids.

We investigate lasing threshold in random samples created by another positioning algorithm. In order to determine the center of the cylinder, we used the coordinate of the center \((x_r, y_r)\)

![Fig. 16. Lasing thresholds versus normalized disorder index \( |\Delta x_r|_{\text{max}}/a \) (5 samples with another positioning algorithm).](image-url)

![Fig. 15. Average values of \( |\Delta x_r|/|\Delta x_r|_{\text{max}} \) for each \( |\Delta x_r|_{\text{max}} \).](image-url)
directly instead of using \((|\Delta x_r|, \theta)\). Again, random numbers of uniform distribution are used to determine the values of \((x_r, y_r)\) so that the generated cylinder is located with the circular are bounded by the certain radius \(|\Delta r|_{\text{max}}\). The number of samples computed is five. We show the results for lasing threshold in Fig. 16, in which a similar behavior of lasing threshold is observed. We find the tendency in the relation between the lasing threshold and the amount of disorder is independent of positioning algorithms.

5. Conclusion

Investigations of lasing phenomena and lasing thresholds of disordered structures are presented based on FE analyses by changing the disorder index. The threshold of the laser action in the periodic structure is extremely low because of the zero-group velocity at the band edge frequency. However, the lasing phenomena are sensitive to the degrees of disorder, and the threshold of the photonic crystal laser rises by the effect of a small amount of disorder. As the disorder increases, lasing phenomena shift from photonic crystal lasers to random ones, and the threshold of laser action once rises, then it decreases. However, a further increase in the disorder causes the rise of the lasing threshold again. Lasing threshold of random lasing tends to become locally minimum.

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